Exercises

1. Show the following: *A set* $X \subseteq \mathbb{R}$ *is open if and only if for every sequence* x_n *converging to* $a \in A$, $x_n \in A$ *for* n *sufficiently large.*

2. Let $X \subseteq \mathbb{R}$ be open. Show that if $a \in \mathbb{R}$, then $a + X$ is also open, where $a + X = \{a + x; x \in X\}$.

3. Show that $\text{int}(X \cap Y) = \text{int}(X) \cap \text{int}(Y)$, but in general $\text{int}(X \cup Y) \neq \text{int}(X) \cup \text{int}(Y)$. Given an example which illustrates the latter fact.

4. Let A be open and $a \in A$. Show that $A - \{a\}$ is open as well.

5. Show that every collection of nonempty open sets, pairwise disjoints, is countable.

6. Show that the set of accumulation points of a sequence is closed.

7. Let C be closed and $X \subseteq C$. Show that if C is closed then $\overline{X} \subseteq C$.

8. If $\lim x_n = a$ and $X = \{x_1, x_2, \ldots\}$, show that $\overline{X} = X \cup \{a\}$.

9. Let I be a closed interval and suppose $I = A \cup B$, where A, B are closed and disjoints. Show that either $A = I$ or $B = I$.

10. Show that $\frac{1}{4}$ is an element of the Cantor set K. *[Hint: Convince yourself that* $\frac{1}{4}$ *is an accumulation point]*

11. Let $X \subseteq \mathbb{R}$ be countable. Construct a sequence whose accumulation points is the set $\overline{X}$. Use this to show that every closed set is the set of all accumulation points of a sequence. *[Hint: Write* $\mathbb{N}$ *as a countable union of infinite disjoints subsets.]*

12. Let K denote the Cantor set. Show that $[0, 1] = \{|x - y|; x, y \in K\}$. *[Hint: Use the fact that proper fractions whose denominator are power of 3 are dense in* $[0, 1]$*.]*

13. Given any $\alpha > 0$. Show that we can find elements $x_1, x_2, \ldots, x_n$ of the Cantor set such that $\alpha = x_1 + x_2 + \ldots + x_n$. *[Hint: Use exercise 12.]*

14. Show that $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$, but in general $\overline{X \cap Y} \neq \overline{X} \cap \overline{Y}$. Given an example which illustrates the latter fact.

15. Give an example of nested sequence $F_1 \supset F_2 \supset \ldots$ of closed nonempty sets such that $\bigcap_j F_j = \emptyset$.

16. Show that a set X is dense in $\mathbb{R}$ if and only if X^c has empty interior.

17. Give an example of open set A such that $\mathbb{Q} \subseteq A$ and $\mathbb{R} - A$ is uncountable.

18. Given an example of an uncountable closed set containing only transcendental numbers. *[Hint: Use exercise 17.]*

19. Given a nonempty set $X \subseteq \mathbb{R}$ and point $a \in \mathbb{R}$, we define *the distance* of a to X as the number $d(a, X) = \inf\{|x - a|; x \in X\}$. Show that

    1. $d(a, X) = 0 \iff a \in \overline{X}$
    2. If X is closed then we can find $b \in X$ such that $d(a, X) = |a - b|$

20. Show that if X is bounded from above then $\overline{X}$ is as well. Moreover, show that $\sup X = \sup \overline{X}$. Prove the equivalent result for $\inf \overline{X}$.

21. Show that if X is bounded then $\sup X$ and $\inf X$ are adherent points.

22. Show that for every $X \subseteq \mathbb{R}$, the derived set X' is closed.

23. Show that a is an accumulation point of X if and only if it is an accumulation point of $\overline{X}$.

24. Show that $(X \cup Y)' = X' \cup Y'$.

25. Let $X \subseteq \mathbb{R}$ be an open set. Show that every point of X is an accumulation point of X.

26. Let $X \subseteq \mathbb{R}$ be a closed set and $a \in X$. Show that a is an isolated point if and only if $X - \{a\}$ is closed.

27. Explain the meaning of the following sentences. You can't use the words in *italic* in your explanation.

    (a) $a \in X$ is *not* an *interior* point of X;
    (b) $a \in \mathbb{R}$ is *not* an *adherent* point of X;
    (c) $X \subseteq \mathbb{R}$ is *not* an *open* set;
    (d) $X \subseteq \mathbb{R}$ is *not* a *closed* set;
    (e) $a \in \mathbb{R}$ is *not* an *accumulation* point of X;
    (f) $X' = \emptyset$;
    (g) $X \subseteq Y$ but X is *not dense* in Y;
    (h) $int(\overline{X}) = \emptyset$;
    (i) $X \cap X' = \emptyset$;
    (j) $X \subseteq \mathbb{R}$ is *not* a *compact* set;

28. (Lindelof Theorem) Let $X \subseteq \mathbb{R}$. Any open cover of X has a countable subcover.

29. Let $X \subseteq \mathbb{R}$ be an infinite closed countable set. Show that X has infinitely many isolated points.

30. Show that every real number is the limit of a sequence of pairwise disjoint transcendental numbers.

31. Show that if X is uncountable then $X \cap X' \neq \emptyset$.

32. Obtain an open cover of $\mathbb{Q}$ that doesn't have a finite subcover. Do the same for $[0, +\infty)$.

33. Show that the following are equivalent:

    (a) X is bounded;

    (b) Every infinite subset of X has an accumulation point (which could be outside X);

    (c) Every sequence $x_n \in X$ has a convergent subsequence.

34. (Baire Category Theorem) If $X_1, X_2, X_3, \ldots$ are closed sets with empty interior, then their union $\bigcup_{j=1}^{\infty} X_j$ has empty interior. *[Hint: Use the idea of the proof of theorem 96 from the notes]*

35. Show that $\mathbb{Q}$ is not the intersection of a countable collection of open sets.

36. Let $X \subseteq \mathbb{R}$. Show that if X is uncountable then X' is also.

37. Show that for any $X \subseteq \mathbb{R}$, the set $\overline{X} - X'$ is countable.

38. A point $a \in \mathbb{R}$ is called *condensation point* of X, when every open interval containing a, contains uncountable points of X. Let X_c denotes the set of all condensation points. Show that X_c is a *perfect set*, i.e. closed with no isolated points, and that $X - X_c$ is countable.

39. (Bendixson theorem) Every closed set $X \subseteq \mathbb{R}$ can be written as a union of a perfect set and a countable set. *[Hint: Use the exercise 38.]*